Resit Exam — Ordinary Differential Equations (WIGDV–07)

Thursday 2 February 2017, 14.00h–17.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (2 + 8 points)

Consider the following differential equation:

$$y' + 6y - y^2 = 9.$$

(a) Show that this equation has precisely one constant solution.

(b) Compute a solution satisfying the initial condition $y(0) = 2$ and give the largest interval on which the solution is defined.

Problem 2 (3 + 9 points)

Consider the following differential equation:

$$(xy^2 - y)\,dx + x\,dy = 0.$$

(a) Show that this equation is *not* exact.

(b) Use an integrating factor of the form $M(x, y) = \phi(y)$ to solve the equation. Express the solution explicitly as a function of x.

Problem 3 (4 + 10 + 4 points)

Consider the linear equation $\mathbf{y}' = \begin{bmatrix} t^{-1} & -1 \\ t^{-2} & 2t^{-1} \end{bmatrix} \mathbf{y}$, where $t > 0$.

(a) Verify that $\mathbf{y}_1(t) = \begin{bmatrix} t^2 \\ -t \end{bmatrix}$ is a solution.

(b) Compute a second solution of the form

$$\mathbf{y}_2(t) = \phi(t)\mathbf{y}_1(t) + \begin{bmatrix} 0 \\ z(t) \end{bmatrix}.$$

(c) Compute a fundamental matrix $Y(t)$ with the property $Y(1) = I$.

Problem 4 (5 + 7 + 4 + 6 points)

Let $C([0,1])$ denote the linear space of continuous functions $y : [0,1] \to \mathbb{R}$. This space becomes a Banach space under the norm

$$\|y\| = \sup_{x\in[0,1]} |y(x)|e^{-\alpha x}, \qquad \alpha > 0.$$

Consider the integral operator

$$T : C([0,1]) \to C([0,1]), \qquad (Ty)(x) = 1 + \int_0^x \log(1 + y(t)^2)\, dt.$$

Prove the following statements:

(a) $\big|\log(1 + y^2) - \log(1 + z^2)\big| \le \big|y - z\big| \qquad \forall\, y, z \in \mathbb{R}.$

(b) $\big|(Ty)(x) - (Tz)(x)\big| \le \dfrac{e^{\alpha x} - 1}{\alpha}\|y - z\| \qquad \forall\, y, z \in C([0,1]),\ x \in [0,1].$

(c) $\|Ty - Tz\| \le \dfrac{1}{\alpha}\|y - z\| \qquad \forall\, y, z \in C([0,1]).$

(d) The initial value problem

$$y' = \log(1 + y^2), \qquad y(0) = 1.$$

has a unique solution on the interval $[0,1]$.

Problem 5 (3 + 4 + 3 points)

Let $g(x)$ be a continuous function and consider the following 2nd order equation:

$$x^2u'' - 4xu' + 6u = g(x), \qquad x > 0.$$

(a) Find solutions of the homogeneous equation of the form $u(x) = x^\lambda$.

(b) Verify that a particular solution is given by

$$u_p(x) = x^3 \int_1^x \frac{g(t)}{t^4}\, dt - x^2 \int_1^x \frac{g(t)}{t^3}\, dt.$$

(c) Compute a solution that satisfies $u(1) = 1$ and $u'(1) = 4$.

Problem 6 (10 + 3 + 5 points)

Consider the following semi-homogeneous boundary value problem:

$$u'' + u = f(x), \qquad u(0) = 0, \qquad u'(\pi) = 0.$$

(a) Compute Green's function $\Gamma(x, \xi)$.

(b) Sketch the graph of $\Gamma(x, \xi)$ as a function of x for $\xi = \frac{1}{2}\pi$.

(c) Use Green's function to solve the boundary value problem with $f(x) = 1$.

End of test (90 points)

Solution of Problem 1 (2 + 8 points)

(a) If y is a constant solution, then $y' = 0$ so that $6y - y^2 = 9$, or equivalently, $(y-3)^2 = 0$. Hence, $y(x) \equiv 3$ is the only constant solution.
(2 points)

(b) **Method 1: separation of variables.** rewriting the differential equation as

$$y' = (y-3)^2$$

we can solve the equation using separation of variables:

$$\int \frac{1}{(y-3)^2}\,dy = \int dx \quad \Rightarrow \quad -\frac{1}{y-3} = x + C \quad \Rightarrow \quad y = 3 - \frac{1}{x+C}.$$

(4 points)

The initial condition $y(0) = 2$ gives $C = 1$.
(2 points)

The maximal interval of existence is $(-1, \infty)$.
(2 points)

Method 2: Riccati's method. The function $u = y - 3$ satisfies the following Bernoulli equation:

$$u' = u^2.$$

This equation be solved directly using separation of variables. Alternatively, the new variable $z = 1/u$ satisfies the following linear equation:

$$z' = -1.$$

Solving gives

$$z = C - x \quad \Rightarrow \quad u = \frac{1}{C-x} \quad \Rightarrow \quad y = 3 + \frac{1}{C-x}$$

(4 points)

The initial condition $y(0) = 2$ gives $C = -1$.
(2 points)

The maximal interval of existence is $(-1, \infty)$.
(2 points)

Solution of Problem 2 (3 + 9 points)

(a) Define the functions $g(x,y) = xy^2 - y$ and $h(x,y) = x$. Then $g_y = 2xy - 1$ and $h_x = 1$. Since $g_y \neq h_x$ it follows that the equation is not exact.
(3 points)

(b) The function $\phi(y)$ is an integrating factor if and only if

$$\frac{\partial}{\partial y}\left[\phi(y)(xy^2 - x)\right] - \frac{\partial}{\partial x}\left[\phi(y)x\right] = 0,$$

or, equivalently,

$$\phi'(y)(xy^2 - y) + (2xy - 1)\phi(y) - \phi(y) = 0 \quad \Leftrightarrow \quad \phi'(y) = -\frac{2}{y}\cdot\phi(y).$$

Clearly, a solution is given by $\phi(y) = 1/y^2$.
(3 points)

After multiplying the differential equation by $\phi(y)$ it reads as

$$\left(x - \frac{1}{y}\right)dx + \frac{x}{y^2}\,dy = 0.$$

Define a potential function by

$$F(x,y) = \int\left(x - \frac{1}{y}\right)dx + C(y) = \frac{x^2}{2} - \frac{x}{y} + C(y).$$

This function should also satisfy

$$F_y = \frac{x}{y^2} \quad \Rightarrow \quad \frac{x}{y^2} + C'(y) = \frac{x}{y^2}.$$

We can choose $C(y) = 0$.
(3 points)

Finally, the solution is given by

$$F(x,y) = K \quad \Leftrightarrow \quad \frac{x^2}{2} - \frac{x}{y} = K \quad \Leftrightarrow \quad y = \frac{2x}{x^2 - 2K}.$$

(3 points)

Solution of Problem 3 (4 + 10 + 4 points)

(a) We have

$$A(t)\mathbf{y}_1 = \begin{bmatrix} t^{-1} & -1 \\ t^{-2} & 2t^{-1} \end{bmatrix} \begin{bmatrix} t^2 \\ -t \end{bmatrix} = \begin{bmatrix} 2t \\ -1 \end{bmatrix} = \mathbf{y}_1'$$

which shows that $\mathbf{y}_1$ satisfies the homogeneous differential equation.
(4 points)

(b) Compute a second solution of the homogeneous equation of the form

$$\mathbf{y}_2(t) = \phi(t)\mathbf{y}_1(t) + \begin{bmatrix} 0 \\ z(t) \end{bmatrix}.$$

On the one hand we have that

$$\mathbf{y}_2' = \phi'\mathbf{y}_1 + \phi\mathbf{y}_1' + \begin{bmatrix} 0 \\ z' \end{bmatrix} = \phi'\mathbf{y}_1 + \phi A\mathbf{y}_1 + \begin{bmatrix} 0 \\ z' \end{bmatrix}.$$

On the other hand we should have that

$$\mathbf{y}_2' = A\mathbf{y}_2 = \phi A\mathbf{y}_1 + A\begin{bmatrix} 0 \\ z \end{bmatrix}.$$

Therefore we must have

$$\begin{bmatrix} 0 \\ z' \end{bmatrix} = A\begin{bmatrix} 0 \\ z \end{bmatrix} - \phi'\mathbf{y}_1,$$

or, equivalently,

$$\begin{aligned} 0 &= -z - t^2\phi' \\ z' &= 2t^{-1}z + t\phi' \end{aligned}$$

(5 points)

Eliminating ϕ' gives

$$z' = t^{-1}z \quad \Rightarrow \quad z = t.$$

Solving for ϕ gives

$$\phi' = -t^{-1} \quad \Rightarrow \quad \phi = -\log t.$$

Hence, the second solution is given by

$$\mathbf{y}_2 = -\log t \begin{bmatrix} t^2 \\ -t \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} -t^2\log t \\ t(1+\log t) \end{bmatrix}$$

(5 points)

(c) Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent a fundamental matrix is given by

$$\widetilde{Y}(t) = \begin{bmatrix} t^2 & -t^2\log t \\ -t & t(1+\log t) \end{bmatrix}.$$

Multiplying a fundamental matrix on the right side with an invertible matrix gives again a fundamental matrix. In particular,

$$Y(t) := \widetilde{Y}(t)\widetilde{Y}(1)^{-1} = \begin{bmatrix} t^2 & -t^2\log t \\ -t & t(1+\log t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} t^2(1-\log t) & -t^2\log t \\ t\log t & t(1+\log t) \end{bmatrix}$$

is a fundamental matrix that satisfies $Y(1) = I$.
(4 points)

Solution of Problem 4 (5 + 7 + 4 + 6 points)

(a) If $y < z$, then by the Mean Value Theorem there exists $c \in (y, z)$ such that

$$\log(1 + y^2) - \log(1 + z^2) = \frac{2c}{1 + c^2}(y - z).$$

Taking absolute values gives

$$|\log(1 + y^2) - \log(1 + z^2)| = \frac{2|c|}{1 + c^2}|y - z|.$$

(3 points)

Note that

$$0 \le (1 - |c|)^2 = 1 - 2|c| + c^2 \quad \Rightarrow \quad 2|c| \le 1 + c^2 \quad \Rightarrow \quad \frac{2|c|}{1 + c^2} \le 1,$$

which gives the desired inequality.
(2 points)

Note: the last inequality can also be obtained by computing the maximum and minimum of the function $f(t) = 2t/(1 + t^2)$.

(b) Let $y, z \in C([0, 1])$ and $x \in [0, 1]$ be arbitrary. Then

$$\begin{aligned}
\big|(Ty)(x) - (Tz)(x)\big| &= \left|\int_0^x \log(1 + y(t)^2) - \log(1 + z(t)^2)\, dt\right| \\
&\le \int_0^x |\log(1 + y(t)^2) - \log(1 + z(t)^2)|\, dt \\
&\le \int_0^x |y(t) - z(t)|\, dt \\
&= \int_0^x |y(t) - z(t)| e^{-\alpha t} e^{\alpha t}\, dt \\
&\le \|y - z\| \int_0^x e^{\alpha t}\, dt \\
&= \frac{e^{\alpha x} - 1}{\alpha}\|y - z\|
\end{aligned}$$

(7 points)

(c) By part (b) we get

$$\big|(Ty)(x) - (Tz)(x)\big|e^{-\alpha x} \le \frac{1 - e^{-\alpha x}}{\alpha}\|y - z\| \le \frac{1}{\alpha}\|y - z\|.$$

(2 points)

Therefore, we have

$$\|Ty - Tz\| = \sup_{x \in [0,1]} \big|(Ty)(x) - (Tz)(x)\big|e^{-\alpha x} \le \frac{1}{\alpha}\|y - z\|.$$

(2 points)

(d) First we recall Banach's fixed point theorem. Let D be a closed, nonempty subset in a Banach space B. Let the operator $T : D \to B$ map D into itself, i.e., $T(D) \subset D$, and assume that T is a contraction: there exists a number $0 < q < 1$ such that

$$\|Tx - Ty\| \leq q\|x - y\|, \qquad \forall\, x, y \in D,$$

Then the fixed point equation $Tx = x$ has precisely one solution $\bar{x} \in D$.
(3 points)

We take $D = B = C([0, 1])$ and we let $T : B \to B$ be as defined above. Part (b) shows that T is a contraction for $\alpha > 1$ (we can take $q = \frac{1}{\alpha}$). Therefore, all the assumptions of Banach's fixed point theorem are satisfied. This implies that T has a unique fixed point. Noting that

$$Ty = y \quad \Leftrightarrow \quad y(x) = 1 + \int_0^x \log(1 + y(t)^2)\, dt \quad \Leftrightarrow \quad y' = \log(1 + y^2), \quad y(0) = 1$$

completes the proof.
(3 points)

Solution of Problem 5 (3 + 4 + 3 points)

(a) If $u(x) = x^\lambda$, then we find the characteristic equation

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Leftrightarrow \quad (\lambda - 2)(\lambda - 3) = 0.$$

of which the solutions are obviously $\lambda = 2$ and $\lambda = 3$. Hence, $u = x^2$ and $u = x^3$ are solutions of the homogeneous equation.
(3 points)

(b) Differentiating once gives

$$u_p'(x) = 3x^2 \int_1^x \frac{g(t)}{t^4}\, dt - 2x \int_1^x \frac{g(t)}{t^3}\, dt$$

(2 points)

Differentiating once more gives

$$u_p''(x) = 6x \int_1^x \frac{g(t)}{t^4}\, dt - 2 \int_1^x \frac{g(t)}{t^3}\, dt + \frac{g(x)}{x^2}$$

(2 points)

Therefore, it follows that

$$x^2 u_p'' - 4x u_p' + 6u_p = g(x).$$

(c) The general solution is given by

$$u(x) = u_h(x) + u_p(x) = c_1 x^2 + c_2 x^3 + x^3 \int_1^x \frac{g(t)}{t^4}\, dt - x^2 \int_1^x \frac{g(t)}{t^3}\, dt.$$

The initial conditions give

$$c_1 + c_2 = 1, \qquad 2c_1 + 3c_2 = 4,$$

which implies that $c_1 = -1$ and $c_2 = 2$.
(3 points)

Solution of Problem 6 (10 + 3 + 5 points)

(a) First we solve the homogeneous differential equation:

$$u'' + u = 0 \quad \Rightarrow \quad u(x) = c_1 \cos(x) + c_2 \sin(x).$$

(2 points)

The solution $u_1(x) = \sin(x)$ satisfies the left boundary condition $u(0) = 0$.
(2 points)

The solution $u_2(x) = \cos(x)$ satisfies the right boundary condition $u'(\pi) = 0$.
(2 points)

Their Wronskian determinant is

$$W = u_1 u_2' - u_1' u2 = -1.$$

(2 points)

Since $p(x) \equiv 1$ the Green's function is given by

$$\Gamma(x, \xi) = \begin{cases} -\sin(\xi)\cos(x) & \text{if } 0 \leq \xi \leq x \leq \pi, \\ -\sin(x)\cos(\xi) & \text{if } 0 \leq x \leq \xi \leq \pi. \end{cases}$$

(2 points)

(b) We have

$$\Gamma(x, \tfrac{1}{2}\pi) = \begin{cases} -\cos(x) & \text{if } \frac{1}{2}\pi \leq x \leq \pi, \\ 0 & \text{if } 0 \leq x \leq \frac{1}{2}\pi. \end{cases}$$

For $0 \leq x \leq \frac{1}{2}\pi$ the we have to draw the graph of the zero function.
(1 point)

For $\frac{1}{2}\pi \leq x \leq \pi$ we have to draw the graph of $-\cos(x)$.
(2 points)

(c) In general we have

$$u(x) = \int_0^\pi \Gamma(x, \xi) f(\xi) d\xi.$$

(2 points)

In particular, for $f(x) = 1$ we have

$$\begin{aligned} u(x) &= \int_0^\pi \Gamma(x, \xi) f(\xi) d\xi \\ &= -\cos(x) \int_0^x \sin(\xi) d\xi - \sin(x) \int_x^\pi \cos(\xi) d\xi \\ &= -\cos(x)(1 - \cos(x)) - \sin(x)(\sin(\pi) - \sin(x)) \\ &= 1 - \cos(x) \end{aligned}$$

(3 points)